Supplementary matterial

1 Parametric approximation to posterior

In this section we outline a variational argument in favour of our *Gaussian* $\times$ *Beta* approximation to the true posterior. The probability distribution of a sensor depth measurement given the true depth Z and the inlier ratio π according to our mixture model is given by:

$$p\left(x_n|Z,\pi\right) = \pi N\left(x_n|Z,\tau_n^2\right) + (1-\pi)\,U\left(x_n\right). \tag{1}$$

We also assume there is a prior distribution for Z and π such that without any other information these quantities are probabilisticaly independent. I.e.

$$p\left(Z,\pi\right) = p\left(Z\right)p\left(\pi\right). \tag{2}$$

Let us augment the model with binary latent variables $y_1 \ldots y_N$ such that

$$p\left(x_n|Z,\pi,y_n\right) = N\left(x_n|Z,\tau_n^2\right)^{y_n} U\left(x_n\right)^{1-y_n} \tag{3}$$

and

$$p\left(y_n|\pi\right) = \pi^{y_n}\left(1-\pi\right)^{1-y_n}. \tag{4}$$

A value of $y_n = 1$ indicates that the n-th measurement was an inlier while if $y_n = 0$ that measurement was an outlier. When these latent variables are marginalised out of the model described by (3) and (4) we get back to the simple mixture model of (1). Define $\mathcal{X} = (x_1, \ldots, x_N)$ and $\mathcal{Y} = (y_1, \ldots, y_N)$. Then the full joint distribution is given by

$$p\left(\mathcal{X}\mathcal{Y},Z,\pi\right) = \left[\prod_{n=1}^{N} p\left(x_n|Z,\pi,y_n\right)p\left(y_n|\pi\right)\right]p\left(Z\right)p\left(\pi\right). \tag{5}$$

Let $q\left(\mathcal{Y},Z,\pi\right)$ be our approximation to the posterior $p\left(\mathcal{Y},Z,\pi|\mathcal{X}\right)$. We will assume that it satisfies the following factorization property:

$$q\left(\mathcal{Y},Z,\pi\right) = q_{\mathcal{Y}}\left(\mathcal{Y}\right)q_{Z,\pi}\left(Z,\pi\right). \tag{6}$$

We now proceed by looking for an approximating distribution $q\left(\mathcal{Y},Z,\pi\right)$ that minimises its KL divergence from the true posterior. It can be shown (section 10.1.1 of **(author?)** [1]) that such a distribution must satisfy the properties:

$$\ln q_{Z,\pi}\left(Z,\pi\right) = E_{\mathcal{Y}}\left[\ln p\left(\mathcal{X},\mathcal{Y},Z,\pi\right)\right] + const. \tag{7}$$

and

$$\ln q_{\mathcal{Y}}\left(Z,\pi\right) = E_{Z,\pi}\left[\ln p\left(\mathcal{X},\mathcal{Y},Z,\pi\right)\right] + const. \tag{8}$$

where the expectation operators $E_{\mathcal{Y}}$ and $E_{Z,\pi}$ denote expectations over the distributions $q_{\mathcal{Y}}\left(\mathcal{Y}\right)$ and $q_{Z,\pi}\left(Z,\pi\right)$ respectively. Now equation (8) leads to

$$\begin{aligned}\ln q_{Z,\pi}(Z,\pi) &= \sum_{n=1}^{N} E_{\mathcal{Y}}[y_n]\left(\ln N\left(x_n|Z,\tau_n^2\right)+\ln\pi\right)\\ &+\sum_{n=1}^{N}\left(1-E_{\mathcal{Y}}[y_n]\right)\left(\ln U(x_n)+\ln(1-\pi)\right)\\ &+\ln p(Z)+\ln p(\pi)+const.\end{aligned} \tag{9}$$

By exponentiating both sides of (9) we get

$$q_{Z,\pi}(Z,\pi)=\left[\prod_{n=1}^{N} N\left(x_n|Z,\tau_n^2\right)^{r_n}\right]\pi^S(1-\pi)^{N-S}p(Z)\,p(\pi). \tag{10}$$

where we have defined

$$r_n = E_{\mathcal{Y}}[y_n] \tag{11}$$

and

$$S=\sum_{n=1}^{N} r_n. \tag{12}$$

If we select convenient conjugate priors for Z and π, (e.g. a Gaussian prior for Z and a uniform for π) then we can recognise our approximating distribution (10) as having the $Gaussian \times Beta$ form.

2 Parametric posterior update formulae

Our approximation to the true posterior has the form

$$q\left(Z,\pi|a,b,\mu,\sigma^2\right) := N\left(Z|\mu,\sigma^2\right)Beta(\pi|a,b) \tag{13}$$

where $N\left(Z|\mu,\sigma^2\right)$ is the Gaussian distribution and

$$Beta(\pi|a,b)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\pi^{a-1}(1-\pi)^{b-1} \tag{14}$$

with Γ denoting the Gamma function. Our update step is based on matching the first and second moments in Z and π of

$$p(x|Z,\pi)\,q\left(Z,\pi|a,b,\mu,\sigma^2\right) \tag{15}$$

with those of

$$q\left(Z,\pi|a',b',\mu',\sigma'^2\right) \tag{16}$$

where we have dropped the n and $n-1$ subscripts to unclutter the notation. Using the moment constraints we compute the new posterior parameters a',b',μ',σ'^2 from the old parameters a,b,μ,σ^2 and the new depth measurement x.

Substituting the likelihood $p(x|Z,\pi)$ into (15) we get

$$\left(\pi N\left(x|Z,\tau^2\right)+(1-\pi)U(x)\right)N\left(Z|\mu,\sigma^2\right)Beta(\pi|a,b). \tag{17}$$

Using the definition of the Beta distribution (14) we can rearrange (17) as

$$\begin{gathered}\tfrac{a}{a+b}N\left(x|\mu,\sigma^2+\tau^2\right)N\left(Z|m,s^2\right)Beta(\pi|a+1,b)\\ +\tfrac{b}{a+b}U(x)\,N\left(Z|\mu,\sigma^2\right)Beta(\pi|a,b+1)\end{gathered} \tag{18}$$

where we have defined

$$\frac{1}{s^2} = \frac{\mu}{\sigma^2} + \frac{x}{\tau^2} \tag{19}$$

and

$$m = s^2 \left(\frac{\mu}{\sigma^2} + \frac{x}{\tau^2} \right). \tag{20}$$

Now if we define

$$C_1 = \frac{a}{a+b} N \left(x | \mu, \sigma^2 + \tau^2 \right) \tag{21}$$

and

$$C_2 = \frac{b}{a+b} U \left(x \right) \tag{22}$$

then the by equating the first and second moments w.r.t. Z of (16) and (18) we get:

$$\mu' = C_1 m + C_2 \mu \tag{23}$$

and

$$\sigma'^2 + \mu'^2 = C_1 \left(s^2 + m^2 \right) + C_2 \left(\sigma^2 + \mu^2 \right). \tag{24}$$

Equating the first and second moments w.r.t. π of (16) and (18) gives:

$$\frac{a'}{a'+b'} = C_1 \frac{a+1}{a+b+1} + C_2 \frac{a}{a+b+1} \tag{25}$$

and

$$\begin{aligned} \frac{a' \left(a'+1 \right)}{\left(a'+b' \right) \left(a'+b'+1 \right)} &= C_1 \frac{\left(a+1 \right) \left(a+2 \right)}{\left(a+b+1 \right) \left(a+b+2 \right)} \\ &\quad + C_2 \frac{a \left(a+1 \right)}{\left(a+b+1 \right) \left(a+b+2 \right)}. \end{aligned} \tag{26}$$

Solving the system of equations (23)(24)(25)(26) we obtain the new posterior parameters a', b', μ', σ'^2 from the old parameters a, b, μ, σ^2 and the new measurement x.

References

[1] Christopher M. Bishop. *Pattern Recognition and Machine Learning (Information Science and Statistics)*. Springer, 1 edition, October 2007.